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On Quantization of Non-Linear Relativistic Field Without Recourse to Perturbation Theory

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Abstract

A new method of quantization of non-linear relativistic field based on separation of the concepts of energy and Hamiltonian is suggested. The proposed method allows the making of the vacuum a stationary state with zero eigenvalues of energy, momentum and charge, and to obtain the exact solutions without recourse to perturbation theory. Infinite integrals, if they appear, turn to be in the denominator and cause no difficulties. An application of this method is demonstrated by an example of a non-trivial relativistic model. Mass-spectrum and S-matrix have been obtained. Only exact solutions are used.

In the present paper the different-time commutation relations for complex scalar field are calculated. The field satisfies a non-linear Lorentz-invariant equation. The calculation is made without the use of perturbation theory. It is possibly due to that, that the subject of quantization is not particle but world-line (emlon).† The number of emlons is conserved in the model considered. It permits us to separate the problem of quantization into parts: for example, one-emlon problem, two-emlon problem, and so on, and to solve each problem independently without using the perturbation theory.

The method is taken from non-relativistic quantum mechanics, where particle number conservation permits separation of the problem into one-particle problem, two-particle problem, and so on, and each problem can be solved independently.

The particle‡ and antiparticle are different states of the emlon in a such description. The idea is not new. It was advanced by Stueckelberg (1942)

† Later on, for briefness the word 'emlon' instead of 'world-line' will be used. Emlon is an abbreviation (ML) from the Russian word 'world-line'.

‡ For the solving of raised problems (calculation of the commutation relations) the notions of particle and antiparticle need not be introduced. In the present article they are not defined, and are used in the intuitive sense.

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and by Feynman (1949). The understanding that energy and Hamiltonian are different quantities by description of particles and antiparticles in terms of world-lines is new and essential (Rylov, 1970). The identification of energy with Hamiltonian contradicts the idea of particle and antiparticle as being different states of the emlon. This identification is consistent only with the consideration of particle and antiparticle as two different subjects.

In the first section of this article the statement of problem is formulated. In the second one the commutation relation for the one-emlon problem is calculated. The commutation relation for two-emlon problem is calculated in the third and fourth sections. The S-matrix is calculated in the last section.

1. *The Statement of the Problem*

Designation. Let x^i be coordinates in $(n+1)$-dimensional space-time.

The cases $n = 1, 2, 3$ will be considered.

Vectors in space-time are denoted by a lower-case letter or by a letter with a Latin index, or even by the whole expression (for instance, $k, x^i, \mathscr{P}_i, \Lambda x$) and vectors in configurational space are denoted by bold type or by a letter with a Greek index (for instance, $\mathbf{k}$, x^α, $\mathscr{P}_\alpha$). In the cases when a vector is taken as the expression, its components are denoted by an index near the round brackets in which the expression is put. 0 denotes the time component, a Greek index the spatial one, and a Latin index spatial and time components together. The superscripts and subscripts denote contravariant and covariant components, separately. The Latin indices take the values $0, 1, \ldots, n$; the Greek ones take $1, 2, \ldots, n$. As usual, summation is made on like super- and subscripts.

Lowering and raising of the indices is accomplished by means of the metric tensor $g_{ik} = g_i \delta_{ik}$ (there is no summation) $g_0 = 1$, $g_\alpha = -1$.

The scalar field described by Lagrangian density

$$L = \varphi_i^+ \varphi^i - m^2 \varphi^+ \varphi + \frac{\lambda}{2} \varphi^+ \varphi^+ \varphi\varphi$$

$$\varphi = \varphi(x), \qquad \varphi_i = \partial_i \varphi, \qquad \varphi^i = \partial^i \varphi, \qquad x = \{t, \mathbf{x}\} \tag{1.1}$$

is considered.

$\varphi(x)$ is an operator in Hilbert space $\mathscr{H}$ of vectors ϕ. $\varphi^+(x)$ is the Hermitian conjugate operator. λ is a constant. As well as in non-relativistic quantum mechanics, it is taken that $\varphi(x)$ contains only annihilation operators, i.e.

$$\varphi(x)\, \phi_0 \equiv \varphi(x)|0\rangle = 0 \tag{1.2}$$

where ϕ_0 is a vacuum vector.

Equation (1.2) is considered (Berestetskij *et al.*, 1968, Section 11) to be impermissible in relativistic theory. As a matter of fact, (1.2) is not at variance with relativity. It is incompatible only with identification of energy and Hamiltonian. Let the physical quantities energy E and momentum $\mathscr{P}_\alpha$ be defined by

$$E=\int T_0{}^0\,d\mathbf{x},\qquad \mathscr{P}_\alpha=\int T_\alpha{}^0\,d\mathbf{x},\qquad d\mathbf{x}=d\mathbf{x}^1\,dx^2\ldots dx^n$$

$$T_i{}^k=\varphi_i{}^+\varphi^k+\varphi^{+k}\varphi_i-\delta_i{}^kL,\qquad \varphi^i\equiv\partial^i\varphi,=g^{ik}\varphi_k \tag{1.3}$$

The dynamical quantities Hamiltonian H and canonical momentum π_α [according to Rylov (1970) canonical momentum π_α is different to momentum $\mathscr{P}_\alpha$] are defined by

$$\partial_0\varphi=\frac{1}{i}[\varphi,H]_-,\qquad \partial_\alpha\varphi=\frac{1}{i}[\varphi,\pi_\alpha]_- \tag{1.4}$$

where $[\ldots]_-$ denotes commutator. Traditional condition of E and H coincidence imposes constraints on operator $\varphi(x)$; namely, it must contain both creation and annihilation operators, i.e. condition (1.2) cannot be fulfilled. I shall not make any assumption about the relation between E and H, but use (1.2). Formally, the method presented is close to the papers of Bogoljubov *et al.* (1958), and Beresin (1965).

Emlon is a subject of quantization. Its state is described by dynamical variable $K=(\epsilon_k,\mathbf{k})$, where $\mathbf{k}$ is a canonical momentum and $\epsilon_k=\operatorname{sign}k_0$ is a sign of time component of momentum k. A particle and antiparticle are two different states of the emlon. They are distinguished by ϵ_k. A number of emlons (i.e. a flow of world-lines through the hyperplane $t=x^0=\text{const.}$) is conserved. It corresponds to conservation of operator

$$N=\int j^0\,d\mathbf{x},\qquad j^k=-i(\varphi^{+k}\varphi-\varphi^+\varphi^k) \tag{1.5}$$

which is independent on the time due to the field equations

$$(\partial_i\partial^i+m^2)\varphi=\lambda\varphi^+\varphi\varphi \tag{1.6}$$

Let us use the designations

$$kx\equiv k_ix^i=k_0x^0+k_\alpha x^\alpha,\qquad k_0=\epsilon_kE(\mathbf{k})$$

$$E(\mathbf{k})=E(k)=|\sqrt{(\mathbf{k}^2+m^2)}|,\qquad \beta(\mathbf{k})=\beta(k)=2E(\mathbf{k}) \tag{1.7}$$

$$K=\{\epsilon_k,\mathbf{k}\},\qquad \int(.)\,dK=\sum_{\epsilon_k=\pm1}\int(.)\,d\mathbf{k},\qquad d\mathbf{k}=dk\,dk_2\ldots dk_n$$

and introduce Fourier-components $a(K,t)=a(\epsilon_k,\mathbf{k},t)$ of φ by means of the relations

$$\varphi(\mathbf{x},t)=\frac{1}{(2\pi)^{n/2}}\int\exp(-ikx)\frac{a(K,t)}{\sqrt{[\beta(k)]}}\,dK \tag{1.8}$$

$$\dot\varphi(\mathbf{x},t)=\frac{\partial\varphi(\mathbf{x},t)}{\partial t}=-i(2\pi)^{-n/2}\int\frac{\epsilon_k}{2}\sqrt{[\beta(k)]}\exp(-ikx)\,a(K,t)\,dK \tag{1.9}$$

The corresponding expressions for Hermitian conjugate quantities can be easily obtained. Reverse transformation has the form

$$a(K,t)=(2\pi)^{-n/2}\int d\mathbf{x}\exp(ikx)\left[\frac{\sqrt{[\beta(k)]}}{2}\varphi(\mathbf{x},t)+\frac{i\epsilon_k}{\sqrt{[\beta(k)]}}\dot{\varphi}(\mathbf{x},t)\right] \tag{1.10}$$

In terms of $a(K,t)$ the (1.6) takes the form

$$\dot{a}(K,t)=i\lambda\frac{\epsilon_k f(\mathbf{k},t)}{\sqrt{[\beta(\mathbf{k})]}}\exp(ik_0 t) \tag{1.11}$$

where

$$\begin{aligned}f(\mathbf{k},t)=(2\pi)^{-n/2}\int &\frac{\exp\{it[\epsilon_{k_1}E(k_1)-\epsilon_{k_2}E(k_2)-\epsilon_{k_3}E(k_3)]\}}{\sqrt{[\beta(k_1)\beta(k_2)\beta(k_3)]}}\\ &\times a^+(K_1,t)\,a(K_2,t)\,a(K_3,t)\,\delta(\mathbf{k}+\mathbf{k}_1-\mathbf{k}_2-\mathbf{k}_3)\\ &\times dK_1\,dK_2\,dK_3\end{aligned} \tag{1.12}$$

$\varphi(x)$ is a scalar and with translation

$$x^i\to\tilde{x}^i=x^i+a^i,\qquad \tilde{x}=x+a \tag{1.13}$$

it is transformed according to the law

$$\varphi(x)\to\tilde{\varphi}(\tilde{x})=\varphi(x)=\varphi(\tilde{x}-a) \tag{1.14}$$

In the proper Lorentz transformation

$$x^i\to\tilde{x}^i=\Lambda^i_{.k}x^k,\qquad \tilde{x}=\Lambda x,\qquad \Lambda^i_{.k}g^{kl}\Lambda^j_{.l}=g^{ij},\qquad \Lambda^0_{.0}>0 \tag{1.15}$$

$\varphi(x)$ is transformed according to the law

$$\begin{aligned}\varphi(x)\to\tilde{\varphi}(\tilde{x})=\varphi(x)=\varphi(\Lambda^{-1}\tilde{x}),\qquad (\Lambda\Lambda^{-1})^i_{.k}=\delta_k{}^i\\ \varphi_i(x)\to\tilde{\varphi}_i(\tilde{x})=(\Lambda^{-1})^k_{.i}\,\varphi_k(\Lambda^{-1}\tilde{x})\end{aligned} \tag{1.16}$$

Using equations (1.10) (1.14) and (1.16) one can easily obtain the transformation law of $a(k,t)$ with the transformations of (1.13) and (1.15):

$$a(\epsilon,\mathbf{k},t)\to\tilde{a}(\tilde{\epsilon},\mathbf{k},\tilde{t})=a(\tilde{\epsilon},\tilde{\mathbf{k}},\tilde{t}-a^0)\exp(i\tilde{k}a),\qquad \tilde{k}_0=\tilde{\epsilon}E(\mathbf{k}) \tag{1.17}$$

$$\begin{aligned}\frac{\tilde{a}_{\tilde{\epsilon}}(\tilde{\mathbf{k}},\tilde{t})}{\sqrt{[\beta(\tilde{k})]}}&=(2\pi)^{-n}\int d\mathbf{x}\int dk\exp[i(\tilde{k}\tilde{x}-k(\Lambda^{-1}\tilde{x})]\\ &\times M(K,\tilde{K},\Lambda)\frac{a_\epsilon(\mathbf{k},(\Lambda^{-1}\tilde{x})^0)}{\sqrt{[\beta(k)]}}\end{aligned} \tag{1.18}$$

where

$$M(K,\tilde{K},\Lambda)=M(\epsilon_k,\mathbf{k};\epsilon_{\tilde{k}},\mathbf{k},\Lambda)=\frac{1}{2}\left(1+\frac{(k\Lambda^{-1})_0}{\tilde{k}_0}\right) \tag{1.19}$$

$$(k\Lambda^{-1})_0=k_i(\Lambda^{-1})^i_{.0},\ (\Lambda^{-1}\tilde{x})^0\equiv(\Lambda^{-1})^0_{.i}\,\tilde{x}^i=t,\qquad \tilde{k}_0=\epsilon_{\tilde{k}}E(\tilde{\mathbf{k}}) \tag{1.20}$$

The vacuum vector $\phi_0 = |0\rangle$ and conjugate vector $\phi^* = \langle 0|$ are defined by

$$a(K)|0\rangle = 0, \qquad \langle 0|a^+(K) = 0, \qquad a(K) \equiv a(K,0) \tag{1.21}$$

From (1.11) and (1.12) for any t it follows

$$a(K,t)|0\rangle = 0, \qquad \langle 0|a^+(K,t) = 0 \tag{1.22}$$

Hence, due to (1.8), compatibility of the (1.2) with the (1.6) follows. By means of (1.8) and (1.9), (1.3) and (1.5) can be written in the terms of $a(K,t)$

$$N = N(t) = \int \epsilon_k a^+(K,t)\, a(K,t)\, dK$$

$$E = E(t) = \int \left[E(K)\, a^+(K,t)\, a(K,t) - \frac{\lambda}{2} a^+(K,t) \frac{\exp(ik_0 t) f(\mathbf{k},t)}{\sqrt{[\beta(\mathbf{k})]}} \right] dK,$$

$$\mathscr{P}_\alpha = \mathscr{P}_\alpha(t) = \int \epsilon_k k_\alpha\, a^+(K,t)\, a(K,t)\, dK \tag{1.23}$$

From (1.23) and (1.22) it follows that

$$N|0\rangle = 0, \qquad E|0\rangle, \qquad \mathscr{P}_\alpha|0\rangle = 0 \tag{1.24}$$

i.e. in vacuum state a number of emlons, their energy and momentum are equal to zero. It justifies the use of the term vacuum.

Let us write commutation relation as

$$[a(K,t), a^+(K',t')]_- = D(t,t';K,K') \tag{1.25}$$

where $D(t,t';K,K')$ is some operator depending on parameters t, t', K, K'. From (1.25) it follows

$$D(t,t';K,K') = D^+(t',t;K',K) \tag{1.26}$$

where ($^+$) is the sign of the Hermitian conjugate. Let us look for $D(t,t';K,K')$ in the form

$$D(t,t';K,K') = D_0(t,t';K,K') + \int D_1(t,t';K,K';\mathscr{P},\mathscr{P}')$$
$$\times\, a^+(\mathscr{P})\, a(\mathscr{P}')\, d\mathscr{P}\, d\mathscr{P}' + \ldots \tag{1.27}$$

c-numerical functions D_l can be calculated based on the following conditions.

(I) Relation (1.25) is compatible with the equations of motion (1.11).

(II) Relation (1.25) is invariant relative to translations (1.13) and proper Lorentz transformation (1.15).

(III) Norm (ϕ,ϕ) of any non-zero vector of state ϕ is positive

$$(\phi,\phi) > 0 \tag{1.28}$$

(IV) Operator N defined by the (1.5) or (1.23) has only the integer eigenvalues (including the negative). The l-emlon state of the form

$$\int f(K_1,\ldots,K_l)\, a^+(K_1),\ldots,a^+(K_l)|0\rangle\, dK_1\, dK_2,\ldots,dK_l$$

is corresponded by the eigenvalue N' and $|N'| \leqslant l$.

Let us consider in turn all these conditions. By differentiating of the (1.25) with respect to t and t' and substituting the expression (1.11) instead of $\dot{a}(K,t)$ the condition of compatibility of (1.25) with (1.11) can be obtained. The result is

$$\frac{\partial D}{\partial t}(t,t';K,K') = \frac{i\epsilon_k \lambda}{(2\pi)^n}\int dK_1\, dK_2\, dK_3 \exp\{it[\epsilon_k E(\mathbf{k}) + \epsilon_{k_1} E(k_1) - \epsilon_{k_2} E(k_2) - \epsilon_{k_3} E(k_3)]\}\{[a^+(K,t), a^+(K',t')]_- a(K_2,t)\, a(K_3,t) + a^+(K_1,t)\, D(t,t';K_2,K)\, a(K_3,t) + a^+(K_1,t)\, a(K_2,t) \times D(t,t';K_3,K')\}\frac{\delta(\mathbf{k}+\mathbf{k}_1-\mathbf{k}_2-\mathbf{k}_3)}{\sqrt{[\beta(k)\,\beta(k_1)\,\beta(k_2)\,\beta(k_3)]}} \tag{1.29}$$

The expression for $\partial D(t,t';K,K')/\partial t'$ due to (1.26) can be obtained from (1.29) by means of Hermitian conjugate and the substitution $t \leftrightarrow t'$, $K \leftrightarrow K'$.

The condition of relativistic invariancy denotes that if into the relation (1.25) which is written in the transformed coordinate system

$$[\tilde{a}(\tilde{K},\tilde{t}), \tilde{a}^+(\tilde{K}',\tilde{t}')]_- = D_0(\tilde{t},\tilde{t}';\tilde{K},\tilde{K}') + \int D_1(\tilde{t},\tilde{t};\tilde{K},\tilde{K}';\mathscr{P},\mathscr{P}') \times \tilde{a}^+(\mathscr{P})\,\tilde{a}(\mathscr{P}')\,d\mathscr{P}\,d\mathscr{P}' + \cdots \tag{1.30}$$

$\tilde{a}(K,t)$ and $\tilde{a}^+(K,t)$ are substituted by expressing them through $a(K,t)$ and $a^+(K,t)$ by means of (1.17) and (1.18) then as a result due to (1.25) and (1.29) an identity will be obtained.

Conditions III and IV impose additional constraints on the functions D_l.

2. *Calculation of* D_0

For determination of $D_0(t,t';K,K')$ let us take the average over vacuum from (1.29) and analogous expression for $\partial D_0/\partial t'$. It follows, due to (1.22), that

$$\frac{\partial D_0(t,t';K,K')}{\partial t} = 0, \qquad \frac{\partial D_0(t,t';K,K')}{\partial t'} = 0 \tag{2.1}$$

or

$$D_0(t,t';K,K') = D_0(K,K') \tag{2.2}$$

Taking into account relativistic invariance, one obtains

$$D_0(K,K') = A(\epsilon_k)\,\delta(K-K')$$
$$\delta(K-K') \equiv \delta_{\epsilon_k,\epsilon_{k'}}\,\delta(\mathbf{k}-\mathbf{k}') \tag{2.3}$$

The condition of positivity of norm of the vector

$$\phi_1 = \int \psi(k)\, a^+(K)|0\rangle\, dK \tag{2.4}$$

leads to the conditions

$$A(1) > 0, \qquad A(-1) > 0 \tag{2.5}$$

Let ϕ_1 be an eigenvector of operator N. It follows from (1.23) that

$$\epsilon_k A(\epsilon_k)\psi(K) = N'\psi(K) \tag{2.6}$$

where N' is the eigenvalue of operator N. According to the fourth condition $|N'| \leqslant 1$, and from (2.5) and (2.6), it follows that

$$A(\epsilon_k) = 1 \tag{2.7}$$

Having defined an electric charge operator by

$$Q = eN \tag{2.8}$$

where e is the elementary electric charge, one obtains from (2.7) that $|K\rangle \equiv a^+(K)|0\rangle$ is an eigenvector of operators $E, \mathscr{P}_\alpha, Q$ with the eigenvalues $E(\mathbf{k})$, $\epsilon_k k_\alpha$, $e\epsilon_k$. The energy $E(\mathbf{k}) > 0$. Thus $|K\rangle$ describes a particle or antiparticle, depending on the sign of ϵ_k, with energy $E(\mathbf{k})$, momentum $\epsilon_k k_\alpha$

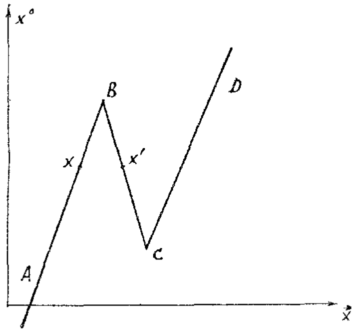

Figure 1

and charge $e\epsilon_k$. One can show that $|K\rangle$ is also an eigenvector of operators H and π_α which are defined by (1.4) with the eigenvalues $\epsilon_k E(\mathbf{k})$ and k_α respectively.

Remark. The function D_0 of the form (2.7) corresponds to the c-numerical part of commutator $[\varphi(x), \varphi^+(x')]_-$. In general it differs from zero for spacelike interval $x - x'$, though it decreases very fast as this interval increases. It is assumed (Pauli, 1947) that the condition

$$[\varphi(x), \varphi^+(x')]_- = 0 \quad \text{for } (x-x')^2 < 0 \tag{2.9}$$

is dictated by the requirement of causality. As a matter of fact in the traditional method of quantization the field variables at the points which are separated by spacelike interval always refer to the different subjects. For example, on the curve $ABCD$ (Fig. 1) (it describes a one-emlon state!) the point x belongs to the particle which is described by the segment AB, and the point x' belongs to antiparticle described by the segment BC. In the traditional method of quantization they are different subjects and describing them variables must commutate. At the suggested approach the whole world-line $ABCD$ is a single subject, hence there is no commutability at the points x and x'. What concerns the causal relation between the events x and x' it exists and is displayed in the fact that with the given choice of the state the particle at the point x and the antiparticle at the point x' will surely be annihilated (formally, it can be seen from the fact that the particle and antiparticle are different states of an emlon). The case when annihilation does not occur is described by the other state with two or more number of emlons.

3. *Calculation of* $D_1(t,t';K,K';\mathscr{P},\mathscr{P}')$

For the calculation of D_1 let us take a matrix element $\langle\mathscr{P}|\cdots|\mathscr{P}'\rangle$ from (1.29). Using the commutation relation (1.25), and the expression (2.3) for D_0 after some calculations the following equation is obtained

$$\frac{\partial D_1}{\partial t}(t,t';K,K';\mathscr{P},\mathscr{P}')=\frac{i\epsilon_k\lambda}{(2\pi)^n} \times\int\frac{\exp\{it[\epsilon_k E(\mathbf{k})+\epsilon_p E(\mathbf{p})-\epsilon_{k_2}E(k_2)-\epsilon_{k_3}E(k_3)]}{\sqrt{[\beta(k)\beta(p)\beta(k_2)\beta(k_3)]}} \times\{D_1(t,t';K_3,K';K_2,\mathscr{P}')+\delta(K_2-K')\,\delta(K_3-\mathscr{P}') +\delta(K_2-\mathscr{P}')\,\delta(K_3-K')\}\,\delta(\mathbf{k}+\mathbf{p}-\mathbf{k}_2-\mathbf{k}_3)\,dK_2\,dK_3 \tag{3.1}$$

Making the substitution

$$D_1(t,t';K,K';,\mathscr{P}\mathscr{P}')=D_1'(t,t';K,K';\mathscr{P},\mathscr{P}')-\delta(\mathscr{P}-K')\,\delta(K-\mathscr{P}') -\delta(\mathscr{P}-\mathscr{P}')\,\delta(K-K') \tag{3.2}$$

and passing from D_1' to its time Fourier-components

$$c(\omega,\omega';K,\mathscr{P};K',\mathscr{P}')=\frac{1}{2\pi}\int\exp[i(\omega-k_0-p_0)t-i(\omega'-k_0'-p_0')t'] \times D_1'(t,t';K,K';\mathscr{P},\mathscr{P}')\,dt\,dt'$$
$$k_0=\epsilon_k E(k),\qquad p_0=\epsilon_p E(p) \tag{3.3}$$

One obtains a homogeneous equation

$$(\omega-k_0-p_0)\,c(\omega,\omega';K,\mathscr{P};K',\mathscr{P}')+\frac{i\lambda\epsilon_k}{(2\pi)^n} \times\int\frac{\delta(\mathbf{k}+\mathbf{p}-\mathbf{k}''-\mathbf{p}'')\,c(\omega,\omega';K'',\mathscr{P}'';K',\mathscr{P}')\,dK''\,d\mathscr{P}''}{\sqrt{\beta(k)\beta(p)\beta(k'')\beta(p'')}}=0 \tag{3.4}$$

Let us pass from the variables $K=\{\epsilon_k,\mathbf{k}\}$, $\mathscr{P}=\{\epsilon_p,\mathbf{p}\}$, $K'=\{\epsilon_{k'},\mathbf{k}'\}$, $\mathscr{P}'=\{\epsilon_{p'},\mathbf{p}'\}$ to the variables

$$\mathbf{s},u=\{\epsilon_k,\epsilon_p,\mathbf{q}\},\qquad \mathbf{s}',u'=\{\epsilon_{k'},\epsilon_{p'},\mathbf{q}'\}\qquad \mathbf{s}=\mathbf{k}+\mathbf{p},$$
$$\mathbf{q}=\mathbf{k}-\mathbf{p},\qquad \mathbf{s}'=\mathbf{k}'+\mathbf{p}',\qquad \mathbf{q}'=\mathbf{k}'-\mathbf{p}' \tag{3.5}$$

One can easily see that

$$\int (.)\, dK d\mathscr{P} = \int (.)\, d\mathbf{s} du, \qquad \int (.)\, du \equiv \frac{1}{2^n} \sum_{\epsilon_k, \epsilon_p = \pm 1} \int (.)\, d\mathbf{q} \tag{3.6}$$

Let us introduce the designations

$$\begin{gathered} \omega(\mathbf{s}, u) = \epsilon_k E(k) + \epsilon_p E(p) \\ \xi_1(\mathbf{s}, u) = \epsilon_k [\beta(\mathbf{k})\beta(\mathbf{p})]^{-1/2}, \qquad \xi_2(\mathbf{s}, u) = \epsilon_k \xi_1(\mathbf{s}, u) \\ d(\omega, \omega'; \mathbf{s}, u; \mathbf{s}', u') = c(\omega, \omega'; K, \mathscr{P}; K', \mathscr{P}') \end{gathered} \tag{3.7}$$

Equation (3.4) is reduced to the form

$$[\omega - \omega(\mathbf{s}, u)]\, d(\omega, \omega'; \mathbf{s}, u; \mathbf{s}', u') + \frac{\lambda}{(2\pi)^n} \int \xi_1(\mathbf{s}, u)\, \xi_2(\mathbf{s}, u'')\, d(\omega, \omega'; \mathbf{s}, u'', \mathbf{s}', u')\, du'' = 0 \tag{3.8}$$

Using with the equation for $\partial D_1/\partial t'$ all the transformations which led from (1.29) to (3.8), one gets

$$[\omega' - \omega(\mathbf{s}', u')]\, d(\omega, \omega'; \mathbf{s}, u; \mathbf{s}', u') + \frac{\lambda}{(2\pi)^n} \int \xi_1(\mathbf{s}, 'u')\, \xi_2(\mathbf{s}', u'')\, d(\omega, \omega'; \mathbf{s}, u; s', u'')\, du'' = 0 \tag{3.9}$$

Thus the determination of function D_1 is reduced to the determination of the function d which satisfies equations (3.8) and (3.9). These equations are essentially identical, because (3.9) is obtained from (3.8), due to the substitution of primed variables by unprimed ones and, conversely, with simultaneous substitution of primed and unprimed arguments of the function d. In (3.8) $\omega', \mathbf{s}'; u', \mathbf{s}$ are parameters, hence essentially one has to deal with the equation

$$[\omega - \omega(u)]\, \chi(u) + \frac{\lambda}{(2\pi)^n} \int \xi_1(u)\, \xi_2(u'')\, \chi(u'')\, du'' = 0 \tag{3.10}$$

Here the argument s in the expressions $\omega(\mathbf{s}, u)$, $\xi_1(\mathbf{s}, u)$, $\xi_2(\mathbf{s}, u)$ is omitted. Further, it will be omitted when it does not lead to misunderstanding.

Let us introduce the following designations

$$I^{(\pm)}(\omega) = \frac{1}{(2\pi)^n} \int \frac{\xi_1(u)\xi_2(u)}{\omega - \omega(u) \pm i\delta}\, du, \qquad \delta \to +0 \tag{3.11}$$

$$\Delta_\pm(u) = 1 + \lambda I^{(\pm)}(\omega(u)) \tag{3.12}$$

Here, and later on, upper or lower sign is taken. One can easily verify that the following functions of argument u are solutions of (3.10)

$$\chi_{\bar{u}}^{(\pm)}(u) = \delta(u - \bar{u}) - \frac{\lambda}{(2\pi)^n} \frac{\xi_1(u)\xi_2(\bar{u})}{[\omega(\bar{u}) - \omega(u) \pm i\delta]\, \Delta_\pm(\bar{u})} \tag{3.13}$$

$$\chi_l(u) = \frac{1}{\sqrt{[(2\pi)^n|I'(\omega_l)|]}} \frac{\xi_1(u)}{(\omega_l - \omega(u))} \tag{3.14}$$

where

$$\bar{u} = \{\epsilon_{\bar{k}}, \epsilon_{\bar{p}}, \bar{\mathbf{q}}\}, \qquad \delta(u-\bar{u}) = \delta_{\epsilon_k,\epsilon_{\bar{k}}}\delta_{\epsilon_p,\epsilon_{\bar{p}}}\delta(\mathbf{q}-\bar{\mathbf{q}}),$$

$$I'(\omega) = \frac{\partial}{\partial\omega} I(\omega) \tag{3.15}$$

and ω_l is one of the roots of the equation†

$$1 + \lambda I(\omega_l) = 0 \tag{3.16}$$

$I(\omega)$ instead of $I^{(\pm)}(\omega)$ is used in the cases when the sign before $i\delta$ in (3.11) has no significance. In reality, $I(\omega)$ depends on s, since s is contained in (3.11) through $\xi_1(\mathbf{s},u)$, $\xi_2(\mathbf{s},u)$ and $\omega(\mathbf{s},u)$. The (3.11) depends on $\omega^2 - \mathbf{s}^2$ only (see Appendix). $I(\omega) = I(\omega,\mathbf{s})$ is an analytical function of ω in the complex plane, with the cuts $(-\infty, -|4m^2 + \mathbf{s}^2|^{1/2}]$, $[|4m^2 + \mathbf{s}^2|^{1/2}, \infty)$ along the real axis. For the values of $M^2 = \omega^2 - \mathbf{s}^2$ from the region $0 < M^2 < 4m^2$, one gets

$$I_1(M) = -\frac{1}{\pi M\sqrt{(4m^2 - M^2)}} \operatorname{arctg} \frac{M}{\sqrt{(4m^2 - M^2)}}, \qquad n = 1 \tag{3.17}$$

$$I_2(M) = -\frac{1}{8\pi M} \ln \frac{2m + M}{2m - M}, \qquad n = 2 \tag{3.18}$$

In (3.17) and (3.18) the principal values of arctangent and logarithm are taken.

For $n = 1$, if $\lambda > 0$ then the equation (3.16) has two roots. If $0 < \lambda < 4\pi m^2$, then $0 < M^2 < 4m^2$. If $\lambda > 4\pi m^2$ then $M^2 < 0$. If $\lambda < 0$, then (3.16) has no roots.

For $n = 2$ if $\lambda > 0$, then equation (3.16) has two roots. If $0 < \lambda < 8\pi m$, then $0 < M^2 < 4m^2$, if $\lambda > 8\pi m$, then $M^2 < 0$. If $\lambda < 0$, (3.16) has no roots.

For $n = 3$ integral $I_3(\omega)$ is infinite, then $\Delta_\pm(\omega) \to \infty$ and the second term in (3.13) vanishes. It is equivalent to $\lambda = 0$. For $n = 3$, equation (3.16) has no roots. Thus in this case a set of functions $\chi_{\bar{u}}^{(+)}(u)$ or $\chi_{\bar{u}}^{(-)}(u)$ coincides with a set of functions for the free field. Though $I_3(\omega)$ is infinite and $\chi_{\bar{u}}^{(+)}(u)$ coincides with $\chi_{\bar{u}}^{(-)}(u)$, by introducing cutoff in integral (3.11) (see Appendix) one can determine $I_3(\omega)$ as an analytical function in complex plane ω with the cuts $(-\infty, -|4m^2 + \mathbf{s}^2|^{1/2}]$, $[|4m^2 + \mathbf{s}^2|^{1/2}, \infty)$ along real axis. The jump of $I_3(\omega)$ on the cut edges will be finite even in the case when cutoff is eliminated and integral diverges. One might use this circumstance for obtaining an S-matrix different from the unit one. For this purpose it is sufficient to carry out renormalization, i.e. to consider a parameter of cutoff as such a function of λ that while $\lambda \to 0$, $I^{(\pm)}(\omega)$ diverges and $\lambda/\Delta_\pm(\bar{u})$ tends to the finite value $\lambda_0 C(\gamma^2)$, where λ_0 is the renormalized interaction constant and $C(\gamma^2)$ is some function of $\gamma^2 = \omega^2 - \mathbf{s}^2$.

† It is assumed that (3.16) has no multiple roots.

Either functions $\chi_{\bar{u}}^{(+)}(u)$, $\chi_l(u)$ and functions $\chi_{\bar{u}}^{(-)}(u)$, $\chi_l(u)$ form a complete set of functions. This can be easily seen if one introduces a conjugate to (3.10)

$$[\omega - \omega(u)]\,\bar{\chi}(u) + \frac{\lambda}{(2\pi)^n}\int \xi_2(u)\,\xi_1(u'')\,\bar{\chi}(u'')\,du'' = 0 \tag{3.19}$$

Its solutions are

$$\bar{\chi}_{\bar{u}}^{(+)}(u) = \delta(u-\bar{u}) - \frac{\lambda}{(2\pi)^n}\frac{\xi_1(\bar{u})\,\xi_2(u)}{[\omega(\bar{u})-\omega(u)\pm i\delta]\,\Delta_{\pm}(\bar{u})} \tag{3.20}$$

$$\bar{\chi}_l(u) = -\frac{\operatorname{sign}(I'(\omega_l))}{\sqrt{[(2\pi)^n|I'(\omega_l)|]}}\frac{\xi_2(u)}{\omega_l - \omega(u)} \tag{3.21}$$

Let us assume that

$$v = \{\bar{u}, l\}, \qquad \chi_v^{(\pm)}(u) = \{\chi_{\bar{u}}^{(\pm)}(u),\ \chi_l(u)\}, \qquad \bar{\chi}_v^{(\pm)}(u) = \{\bar{\chi}_{\bar{u}}^{(\pm)}(u),\ \chi_l(u)\}$$
$$\delta(v-v') = \{\delta(\bar{u}-\bar{u}'),\ \delta_{ll'}\}, \qquad \int(.)\,dv = \int(.)\,d\bar{u} + \sum_l(.) \tag{3.22}$$

then

$$\int \bar{\chi}_v^{(\pm)}(u)\,\chi_{v'}^{(\mp)}(u)\,du = \delta(v-v')$$
$$\int \bar{\chi}_v^{(\pm)}(u)\,\chi_v^{(\mp)}(u')\,dv = \delta(u-u') \tag{3.23}$$

This can be verified by substituting (3.13), (3.14), (3.20) and (3.21) into (3.23), and using (3.11), (3.12) and (3.16) in the calculations.

The solution of (3.8) and (3.9) can be written in the form

$$d(\omega,\omega';\mathrm{s},u;\mathrm{s}',u') = \int dv\,dv'\,\chi_v^{(+)}(u)\,A(\mathrm{s},v;\mathrm{s}',v')\,\chi_{v'}^{(-)}(u')$$
$$\times\,\delta(\omega-\omega(v))\,\delta(\omega'-\omega(v')) \tag{3.24}$$

where $\omega(v) = \{\omega(\bar{u}),\omega_l\}$, $A(\mathrm{s},v;\mathrm{s}',v')$ is an arbitrary function of its arguments.

Let us find the constraints that are imposed on D_1' by condition (II).

Substituting (1.17) into (1.30) and calculating a matrix element $\langle\mathscr{P}|\cdots|\mathscr{P}'\rangle$ one obtains, after transformations,

$$\exp[i(\tilde{k}-\tilde{k}'+p-p')a]\,D_1'(\tilde{t}-a^0,\tilde{t}'-a^0;\tilde{K},\tilde{K}';\mathscr{P},\mathscr{P}')$$
$$= D_1'(\tilde{t},\tilde{t}';\tilde{K},\tilde{K}';\mathscr{P},\mathscr{P}') \tag{3.25}$$

This condition must be true for any translations a. Using (3.3) and (3.7) one sees that in (3.25) $d(\omega,\omega';\mathrm{s},u;\mathrm{s}',u')$ differs from zero only in the case when $\mathbf{s}=\mathbf{s}'$ and $\omega=\omega'$.

Let us introduce the designations

$$\mu_{K,\mathscr{P}}^{(\pm)}(K,\mathscr{P}) = \delta(\bar{\mathrm{s}}-\mathrm{s})\,\chi_{\bar{u}}^{(\pm)}(u), \qquad \sigma = \{\omega,\mathbf{k}+\mathbf{p}\} \tag{3.26}$$

$K, \mathscr{P}$ and $\bar{K}, \bar{\mathscr{P}}$ correlates with s, u and $\bar{\mathrm{s}}$, $\bar{u}$ respectively by means of the (3.5). One can write in new designations

$$c(\sigma,\sigma';K,\mathscr{P};K',\mathscr{P}') = \int \mu_{\bar{K},\bar{\mathscr{P}}}^{(+)}(K,\mathscr{P})\,\delta(\sigma_0-\bar{k}_0-\bar{p}_0)\,\delta(\sigma-\sigma')\,d\bar{K}\,d\bar{\mathscr{P}}$$
$$\times \frac{B(\bar{K},\bar{\mathscr{P}};\bar{K}'.\bar{\mathscr{P}}')}{\sqrt{[\beta(\bar{k})\beta(\bar{p})\beta(\bar{k}')\beta(\bar{p}')]}}\,\delta(\sigma_0'-\bar{k}_0'-\bar{p}_0')$$
$$\times \mu_{\bar{K}',\bar{\mathscr{P}}'}^{(-)}\,d\bar{K}'\,d\bar{\mathscr{P}}' + \delta(\sigma-\sigma')\sum_l \delta(\omega-\omega_l)$$
$$\times \chi_l(k,\mathscr{P})\,B_l(\mathrm{s})\,\chi_l(K',\mathscr{P}') \tag{3.27}$$

Here σ, k, p are vectors in the space-time. δ-Functions containing them have dimensionality $n+1$.

Let us substitute (1.18) into (1.30) and calculate a matrix element $\langle\mathscr{P}|\cdots|\mathscr{P}'\rangle$.

After rather lengthy calculations one obtains the condition which is imposed on $c(\omega,\omega';K,\mathscr{P};K',\mathscr{P}')$ by the relativistic invariance. Substituting (3.27) into it, one obtains the condition.

$$B(\epsilon_k,(k\Lambda)_\alpha;\epsilon_p,(p\Lambda)_\alpha;\epsilon_{k'},(k'\Lambda)_\alpha;\epsilon_{p'},(p'\Lambda)_\alpha)\,\delta(k+p-k'-p')$$
$$= B(\epsilon_k,\mathbf{k};\epsilon_p,\mathbf{p};\epsilon_{k'},\mathbf{k}';\epsilon_{p'},\mathbf{p}')\,\delta(k+p-k'-p') \tag{3.28}$$
$$B_l(\mathrm{s}) = B_l = \text{const.} \tag{3.29}$$

The general solution of (3.28) is an arbitrary function of four invariants ϵ_k, ϵ_p, $\epsilon_{k'}$, $\epsilon_{p'}$ and six scalar products kp, kk', kp', pk', pp', $p'k'$. Because of the presence of δ-function in (3.28) not all invariants mentioned above are independent, and the general solution of (3.28) can be written as a function of three invariants.

$$B = B(K,\mathscr{P};K',\mathscr{P}') = B_1(\epsilon_k,(k+p)^2,(k-k')^2) \tag{3.30}$$

Reversing (3.3) and using (3.27), (3.29) and (3.30) one gets the general form of the function D_1' that satisfies conditions (I) and (II)

$$D_1'(t,t';K,K';\mathscr{P},\mathscr{P}') = \Big\{\int \exp\{-it[\omega(\bar{u})-\omega(u)+i\delta]$$
$$+\,it'[\omega(\bar{u}')-\omega(u')-i\delta]\}\,\chi_{\bar{u}}^{(+)}(u)$$
$$\times B_2(\bar{u},\bar{u}',\mathrm{s})\,\chi_{\bar{u}'}^{(-)}(u')\,d\bar{u}\,d\bar{u}' + \sum_l$$
$$\times \exp\{-i[\omega_l-\omega(u)]\,t + i[\omega_l-\omega(u')]\,t'\}$$
$$\times B_l\,\chi_l(u)\,\chi_l(u')\Big\}\,\delta(\mathrm{s}-\mathrm{s}') \tag{3.31}$$

where

$$B_2(\bar{u},\bar{u}',\mathrm{s}) = \frac{B_1(\epsilon_{\bar{k}},(\bar{k}+\bar{p})^2,(\bar{k}-\bar{k}')^2)}{\sqrt{[\beta(\bar{k})\beta(\bar{p})\beta(\bar{k}')\beta(\bar{p}')]}}\,\delta(\bar{k}_0+\bar{p}_0-\bar{k}_0'-\bar{p}_0') \tag{3.32}$$

and in (3.32) $\mathbf{s} = \bar{\mathbf{k}} + \bar{\mathbf{p}} = \bar{\mathbf{k}}' + \bar{\mathbf{p}}'$.

For determination of the constraints which are imposed by condition (III), let us assume that

$$\phi_2 = \frac{1}{\sqrt{2}} \int dK\, d\mathscr{P} f(K,\mathscr{P})\, a^+(K)\, a^+(\mathscr{P})|0\rangle \tag{3.33}$$

$$f(K,\mathscr{P}) = \sum_l \psi_l(s)\, \bar{\chi}_l(u) + \int d\bar{u}\, \psi(\mathrm{s},\bar{u})\, \bar{\chi}_{\bar{u}}^{(+)}(u) \tag{3.34}$$

Let us calculate (ϕ_2, ϕ_2) using the simultaneous commutation relation, which according to (1.25), (1.27), (2.3) and (3.2) can be written in the form

$$a(K)\, a^+(K') = \delta(K-K')\Big(1 - \int a^+(\mathscr{P})\, a(\mathscr{P})\, d\mathscr{P}\Big) + \int D_1'(K,K';\mathscr{P},\mathscr{P}') \times a^+(\mathscr{P})\, a(\mathscr{P}')\, d\mathscr{P}\, d\mathscr{P}' + O_2 \tag{3.35}$$

where

$$D_1'(K,K';\mathscr{P},\mathscr{P}') = D_1'(0,0;K,K';\mathscr{P},\mathscr{P}') \tag{3.36}$$

and the terms whose contribution into matrix elements between any states ϕ_1 and ϕ_2 defined by equations (2.4) and (3.33) respectively, is equal to zero is designated through O_2.

The calculation gives

$$\begin{aligned}(\phi_2,\phi_2) &= \tfrac{1}{2} \int dK\, dK'\, d\mathscr{P}\, d\mathscr{P}'\, f^*(K,\mathscr{P})\, D_1'(K,K';\mathscr{P},\mathscr{P}') f(K',\mathscr{P}') \\ &= \tfrac{1}{2} \int d\mathrm{s} \Big\{ \sum_l |\psi_l(\mathrm{s})|^2 B_l + \int d\bar{u}\, d\bar{u}'\, B_2(\bar{u},\bar{u}',\mathrm{s}) \\ &\qquad \times \psi^+(\mathrm{s},\bar{u})\, \psi(\mathrm{s},\bar{u}') \Big\}\end{aligned} \tag{3.37}$$

where $f^*(k,\mathscr{P})$ is a complex-conjugate to $f(k,\mathscr{P})$. For the fulfilment of condition (III), it is necessary that (3.37) would not be negative, and vanishes only in the case when $\psi_l(\mathrm{s}) = 0$, $\psi(\mathrm{s},\bar{u}) = 0$.

Let $n = 1$. Let us use the identity

$$\delta(\omega(u) - \omega(u')) = \beta(k)\,\beta(p)\,|\eta\gamma|^{-1}[\delta(u-u') + \delta(u+u')]$$

$$\gamma = \sqrt{[\omega^2(u) - \mathrm{s}^2]}, \qquad \eta = \sqrt{(\gamma^2 - 4m^2)} \tag{3.38}$$

where $u = \{\epsilon_k, \epsilon_p, \mathbf{q}\}$, $-u = \{\epsilon_p, \epsilon_k, -\mathbf{q}\}$ (i.e. $-u$ is derived from u by k and p permutation) and write (3.31) as

$$D_1'(K,K';\mathscr{P},\mathscr{P}') = \delta(\mathrm{s}-\mathrm{s}') \int \chi_v^{(+)}(u)\{B_+(\mathrm{s},v)\, \chi_v^{(-)}(u') + B_-(\mathrm{s},v)\, \chi_{-v}^{(-)}(u')\}\, dv \tag{3.39}$$

where

$$v = \{\bar{u}, l\}, \qquad -v = \{-\bar{u}, l\}$$

$$B_+(\mathrm{s},v) = \{B_+(\mathrm{s},\bar{u}), B_l\}, \qquad B_-(\mathrm{s},v) = \{B_-(\mathrm{s},\bar{u}), B_l\}$$

$$B_+(\mathrm{s},u) = |\eta\gamma|^{-1} B_1(\epsilon_k, \gamma^2, 0), \qquad B_-(\mathrm{s},u) = |\eta\gamma|^{-1} B_1(\epsilon_p, \gamma^2, (k-p)^2) \tag{3.40}$$

If condition (III) is fulfilled for arbitrary functions $\psi(\mathrm{s},\bar{u})$ and $\psi_l(\mathrm{s})$, then

$$B_l > 0, \qquad B_+(\mathrm{s},\bar{u}) > 0, \qquad B_-(\mathrm{s},\bar{u}) = 0 \quad \text{for } \bar{u} \neq -\bar{u} \tag{3.41}$$

Equation (3.41) does not contradict (3.40), because the sets of arguments of function B_1 $(\epsilon_{\bar{k}}, \gamma^2, 0)$ and $(\epsilon_{\bar{p}}, \gamma^2, (\bar{k}-\bar{p})^2)$ coincide only when $\bar{k}=\bar{p}$, but in this case $\bar{u}=-\bar{u}$ and $B_+ = B_-$ does not contradict (3.41).

Assume that (3.39) and (3.40) are true for $n=2,3$. It does not follow necessarily from (3.30) and (3.32), but it also does not contradict them. Equation (3.39) is obtained if one assumes that B_1 in (3.30) differs from zero only when $k=k'$ or $p\cdot k=p'$. Relativistic invariancy of these additional restrictions is evident.

Condition (IV) imposes further restrictions on B_+ and B_-. One gets

$$B_+(\mathrm{s},\bar{u}) + B_-(\mathrm{s},\bar{u}) = 2, \quad B_-(\mathrm{s},\bar{u}) = 0 \quad \text{for } \bar{u} \neq -\bar{u}, \quad B_l = j/2 \tag{3.42}$$

where $j=1$ or $j=2$, according to either a single or a double charge, corresponds to the state $\chi_l(u)$.

Up to now, the emlon identity has not been taken into account. According to the principles of quantum theory the particle identity is described by the fact that the wave function is taken as symmetric or antisymmetric with respect to the identical particles permutation. It seems to be true for emlons too. Let us assume that the wave function $f(K,\mathscr{P})$ in (3.33) is symmetric with respect to the permutation of arguments K and $\mathscr{P}$.

$$f(K,\mathscr{P}) = f(\mathscr{P},K) \tag{3.43}$$

This is shown to be equivalent to the symmetry of function $\psi(\mathrm{s},\bar{u})$ from (3.34)

$$\psi(\mathrm{s},\bar{u}) = \psi(\mathrm{s},-\bar{u}) \tag{3.44}$$

If only symmetric functions (3.43) are taken into account, then condition (III) leads to

$$B_+(\mathrm{s},\bar{u}) + B_-(\mathrm{s},\bar{u}) > 0 \tag{3.45}$$

If only antisymmetric functions $\psi(\mathrm{s},\bar{u})$

$$\psi(\mathrm{s},\bar{u}) = -\psi(\mathrm{s},-\bar{u}) \tag{3.46}$$

are taken into account, then condition (III) leads to inequality

$$B_+(\mathrm{s},\bar{u}) - B_-(\mathrm{s},\bar{u}) > 0 \quad \text{for } \bar{u} \neq -\bar{u} \tag{3.47}$$

Remark. Equation (3.46) is not equivalent to

$$f(K,\mathscr{P}) = -f(\mathscr{P},K) \tag{3.48}$$

and in this sense the symmetric functions (3.44) have the advantage over antisymmetric ones, and the considered model is rather the boson one than fermion one.

Nevertheless, it will be shown that (3.46), but not (3.48), can be imposed, and it leads to the commutation relation with anticommutator.

4. *Calculation of Hamiltonian, Energy and Other Quantities*

Now, operators of physical and dynamical quantities in the space of two emlon states will be calculated. The calculations will be made for the arbitrary functions $f(K,\mathscr{P})$ which are derived from (3.33). Then one will get the restrictions imposed on $B_\pm$ by the requirements of (3.44) or (3.46). According to (1.3), the Hamiltonian is defined by

$$\dot{\mathscr{C}}(K) = i[H,\mathscr{C}(K)]_- \tag{4.1}$$

where instead of $a(K)$ operator $\mathscr{C}(K)$ is used

$$\mathscr{C}(K,t) = \exp[-i\epsilon_k E(\mathbf{k})\,t]\,a(K,t), \qquad \mathscr{C}(K) = \mathscr{C}(K,0) \tag{4.2}$$

Instead of (1.11) one has

$$\dot{\mathscr{C}}(K,t) = -i\epsilon_k E(\mathbf{k})\mathscr{C}(K,t) + \frac{i\lambda\epsilon_k}{\sqrt{[\beta(\mathbf{k})]}} f(\mathbf{k},t) \tag{4.3}$$

$$f(\mathbf{k},t) = \frac{1}{(2\pi)^n}\int dK_1\,dK_2\,dK_3\,\frac{\mathscr{C}^+(K_1,t)\mathscr{C}(K_2,t)\mathscr{C}(K_3,t)}{\sqrt{[\beta(k_1)\beta(k_2)\beta(k_3)]}}\,\delta(\mathbf{k}+\mathbf{k}_1-\mathbf{k}_2-\mathbf{k}_3) \tag{4.4}$$

H is found in the form

$$H = \int H_0(K,\mathscr{P})\mathscr{C}^+(K)\mathscr{C}(\mathscr{P})\,dK\,d\mathscr{P} + \int H_1(K,\mathscr{P};K',\mathscr{P}') \times \mathscr{C}^+(K)\mathscr{C}^+(\mathscr{P})\mathscr{C}(\mathscr{P}')\mathscr{C}(K')\,dK\,d\mathscr{P}\,dK'\,d\mathscr{P}' + O_3 \tag{4.5}$$

where H_0 and H_1 are still unknown c-numerical functions. Their form will be determined from the comparison of the right-hand parts of (4.1) and (4.3). During the calculation of the commutator in (4.1) the simultaneous commutation relation is used. For operators $\mathscr{C}(K)$ it takes the same form (3.35) that for $a(K)$, with the commutation function D_1'

$$D_1'(K,K';\mathscr{P},\mathscr{P}') = \delta(\mathbf{s}-\mathbf{s}')\int \chi_v^{(+)}(u)[B_+(v)\chi_v^{(-)}(u') + B_-(v)\chi_{-v}^{(-)}(u')]\,dv \tag{4.6}$$

$B_+(v)$ and $B_-(v)$ are still unknown functions of $\mathbf{s}$ and v.

Calculation gives

$$H_0(K,\mathscr{P}) = \epsilon_k E(\mathbf{k})\,\delta(K-\mathscr{P}) \tag{4.7}$$

and for H_1 one has

$$\int\Big\{-\int D_1'(K,K'';\mathscr{P},\mathscr{P}'')[H_1(K'',\mathscr{P}'';K',\mathscr{P}') + \epsilon_{k'}E(\mathbf{k}')\,\delta(K''-K') \times \delta(\mathscr{P}''-\mathscr{P}')]\,dK''\,d\mathscr{P}'' + \omega(\mathbf{s},u)\,\delta(K-K')\,\delta(\mathscr{P}-\mathscr{P}') - \frac{\lambda}{(2\pi)^n} \times \xi_1(u)\,\xi_2(u')\,\delta(\mathbf{s}-\mathbf{s}')\Big\} D_1'(K',K''';\mathscr{P}',\mathscr{P}''')\,dK'\,d\mathscr{P}' = 0 \tag{4.8}$$

From (4.8) it is easy to find H_1, but I shall not do it, and use instead (4.8) for calculation of a matrix element $\langle\mathscr{P},K|H|K',\mathscr{P}'\rangle$, where

$$\langle\mathscr{P},K| = \frac{1}{\sqrt{2}}\langle 0|\mathscr{C}(\mathscr{P})\mathscr{C}(K), \qquad |K,\mathscr{P}\rangle = \frac{1}{\sqrt{2}}\mathscr{C}^{+}(K)\mathscr{C}^{+}(\mathscr{P})|0\rangle$$

Let us take a matrix element from (4.5), use the commutation relation and express the obtained combination $D_1'\ H_1\ D_1'$ by means of (4.8). After calculations one obtains

$$\langle\mathscr{P},K|H|K',\mathscr{P}'\rangle = \tfrac{1}{2}\omega(K,\mathscr{P})\,D_1'(K,K';\mathscr{P},\mathscr{P}') - \frac{\lambda}{2.(2\pi)^n}\int \xi_1(u)\,\xi_2(u'') \times \delta(\mathrm{s}-\mathrm{s}'')\,D_1'(K'',K';\mathscr{P}'',\mathscr{P}')\,dK''\,d\mathscr{P}'' \tag{4.9}$$

Let us pass to the other representation, introducing a wave function $\psi(\mathrm{s},v)$ by means of

$$\phi_2 = \int dv\,d\mathrm{s}\,\psi(\mathrm{s},v)|\mathrm{s},v\rangle_+ = \int \bar{\chi}_v^{(+)}(u)\psi(\mathrm{s},v)|K,\mathscr{P}\rangle\,dv\,du\,d\mathrm{s} \tag{4.10}$$

Matrix elements of operator A in (s,v)-representation take the form

$${}_+\langle v,\mathrm{s}|A|\mathrm{s}',v'\rangle_+ = \int \bar{\chi}_v^{(-)}(u)\langle\mathscr{P},K|A|K',\mathscr{P}'\rangle\,\bar{\chi}_{v'}^{(+)}(u')\,du\,du' \tag{4.11}$$

Calculation gives for the Hamiltonian

$${}_+\langle v,\mathrm{s}|H|\mathrm{s}',v'\rangle_+ = \frac{\omega(v)}{2}[B_+(v)\,\delta(v-v') + B_-(v)\,\delta(v+v')]\,\delta(\mathrm{s}-\mathrm{s}')$$

$$\omega(v) = \{\omega(\bar{u}),\omega_l\} \tag{4.12}$$

Using the second equation of (1.4), one can find the form of operator π_α in much the same way as it is made for H. Calculation gives

$${}_+\langle v,\mathrm{s}|\pi_\alpha|\mathrm{s}',v'\rangle_+ = s_\alpha\frac{\delta(\mathrm{s}-\mathrm{s}')}{2}[B_+(v)\,\delta(v-v') + B_-(v)\delta(v+v')] \tag{4.13}$$

Let us introduce the projecting operator Λ_ν, $\nu=-1;1$. Projection to the space $\mathscr{H}_{2,1}$ of symmetric functions $\psi(\mathrm{s},v)$ (3.44) corresponds to $\nu=1$, and projection to the space $\mathscr{H}_{2,-1}$ of antisymmetric functions (3.46) corresponds to $\nu=-1$.

$${}_+\langle v,\mathrm{s}|\Lambda_\nu|\mathrm{s}',v'\rangle_+ = \delta(\mathrm{s}-\mathrm{s}')\,\delta_\nu(v-v'),$$

$$\delta_\nu(v-v') \equiv \tfrac{1}{2}[\delta(v-v') + \nu(v)\delta(v+v')] \tag{4.14}$$

$$v(v) = \{v,\ 1\}$$

One considers the cases of symmetric functions ($\nu=1$) and antisymmetric functions ($\nu=-1$), simultaneously distinguishing them by ν. If $\phi\in\mathscr{H}_{2,\nu}$

and A is a right† operator appropriate to some observable quantity then essentially $A\phi \in \mathscr{H}_{2\nu}$. For this, it is necessary and sufficient that

$$[A, \Lambda_\nu]_- = 0 \tag{4.15}$$

The commutability of H with Λ_ν leads to

$$B_+(v) = B_+(-v), \qquad B_-(v) = B_-(-v) \tag{4.16}$$

Operator N from (1.23) in $(\mathbf{s}, v)$-representation takes the form

$$_+\langle v, \mathbf{s}|N|\mathbf{s}', v'\rangle_+ = \frac{\delta(\mathbf{s}-\mathbf{s}')}{2}\{(\epsilon_v B_+^2 + \epsilon_{-v} B_-^2)\,\delta(v-v') + B_+ B_-(\epsilon_v + \epsilon_{-v})\,\delta(v+v')\} \tag{4.17}$$

where

$$\epsilon_v = \{\epsilon_{\bar{K}}, -\operatorname{sign}(I'(\omega_l))\}, \qquad \epsilon_{-v} = \{\epsilon_{\bar{p}}, -\operatorname{sign}(I'(\omega_l))\}$$

The commutability of N with Λ_ν, together with (3.45) and (3.47), leads to

$$B_+(v) = \nu(v) B_-(v) \tag{4.18}$$

As

$$_+\langle v, \mathbf{s}|\mathbf{s}', v'\rangle_+ = B_+ \frac{\delta(\mathbf{s}-\mathbf{s}')}{2}[\delta(v-v') + \nu(v)\delta(v+v')] \tag{4.19}$$

then condition (IV) (integerness of N) leads to the fact that $2B_+(v)$ can be only integer and positive.

Let us choose

$$B_+(\bar{u}) = \nu B_-(\bar{u}) = 1, \qquad B_l = j/2 \tag{4.20}$$

where $j = 1$ or $j = 2$, depending on either a single or a double charge, is prescribed to the bound state $\psi_l(\mathbf{s})$. Equation (4.20) is true for $n = 1, 2, 3$ but in the case $n = 1$ it is a unique solution and in the cases $n = 2, 3$ the uniqueness is open to questions.

Now, the commutation function is completely determined. One can easily show that in the case when there is no interaction ($\lambda = 0$), such a choice of the commutation function corresponds to

$$\mathscr{C}(K)\mathscr{C}^+(K') - \nu\mathscr{C}^+(K')\mathscr{C}(K) = \delta(K-K') \tag{4.21}$$

Thus,

$$_+\langle v, \mathbf{s}|N|\mathbf{s}', v'\rangle_+ = (\epsilon_v + \epsilon_{-v})\,\delta(\mathbf{s}-\mathbf{s}')\,\delta_\nu(v-v') \tag{4.22}$$

For energy E and momentum $\mathscr{P}_\beta$ one gets, from (1.23),

$$_+\langle v, \mathbf{s}|E|\mathbf{s}', v'\rangle_+ = \frac{\delta(\mathbf{s}-\mathbf{s}')}{2}\int [\chi_v^{(-)}(u) + \nu\chi_{-v}^{(-)}(u)][E(k)\,\delta(u-u') - \frac{\lambda}{2(2\pi)^n}\xi_2(u)\,\xi_2(u')][\chi_{v'}^{(+)}(u') + \nu\chi_{-v'}^{(+)}(u')] \times du\,du' \tag{4.23}$$

† I call the right operator that which can be represented as a sum, each term of which contains an equal number of creation and annihilation operators, and all creation operators are to the left of annihilation ones.

$$
{}_+\langle v, \mathbf{s}|\mathscr{P}_\beta|\mathbf{s}', v'\rangle_+ = \frac{\delta(\mathbf{s}-\mathbf{s}')}{2} \int [\chi_v^{(-)}(u) + \nu\chi_{-v}^{(-)}(u)]\, \epsilon_k k_\beta \times [\chi_{v'}^{(+)}(u') + \nu\chi_{-v'}^{(+)}(u')]\, du' \tag{4.24}
$$

It is easily seen from (4.23) and (4.24) that E and $\mathscr{P}_\beta$ commutate with Λ_ν. Unfortunately, calculation of (4.23) and (4.24) leads to infinite integrals, even when $v \neq v'$.

Nevertheless, (4.23) and (4.24) define operators E and $\mathscr{P}_\beta$. The region of definition of them is a set of functions $\psi(\mathbf{s}, v)$, represented as

$$
\psi(\mathbf{s}, v) = \int \bar{\chi}_v^{(-)}(u)\, g(\mathbf{s}, u)\, du
$$

where $g(\mathbf{s}, u)$ is an arbitrary function with the finite

$$
\int |g(\mathbf{s}, u)|^2 E(k)\, du\, d\mathbf{s}
$$

E, $\mathscr{P}_\beta$ and N are the conserved quantities. This can easily be verified by differentiating (1.23) with respect to time and using equation of motion (1.11). For calculation of $\partial E/\partial t$ the commutation relation is used and at a given stage the energy conservation can be proved only for a single and double emlon states. The commutability of N, E, $\mathscr{P}_\beta$ with π_α is evident from (4.22), (4.23) and (4.24).

The mass square operator M^2 defined by

$$
M^2 = H^2 - \pi_\alpha \pi_\alpha \tag{4.25}
$$

is diagonal in $(\mathbf{s}, v)$-representation

$$
{}_+\langle v, \mathbf{s}|M^2|\mathbf{s}' v'\rangle_+ = M^2(v)\, \delta(\mathbf{s}-\mathbf{s}')\, \delta_\nu(v - v'), \qquad M^2(v) = \omega^2(v) - \mathbf{s}^2 \tag{4.26}
$$

The operator M^2 spectrum, which is naturally called a mass spectrum, consists of a discrete point (with $\lambda > 0$) which is determined from (3.16), (3.17) and (3.18), and of two regions of continuous spectrum, $-s^2 \leqslant M^2 \leqslant 0$, $M^2 \geqslant 4m^2$.

Mass, which is defined by (4.26), is a dynamical quantity in the sense that it is defined through dynamical quantities H, π_α but not through physical quantities $E, \mathscr{P}_\alpha$.

5. *The S-Matrix*

Let

$$
|\mathbf{s}, v\rangle_+ = \int \bar{\chi}_v^{(+)}(u)|K, \mathscr{P}\rangle\, du, \qquad |\mathbf{s}, v\rangle_- = \int \bar{\chi}_v^{(-)}(u)|K, \mathscr{P}\rangle\, du \tag{5.1}
$$

Let us define the S-matrix (Schweber, 1961, formula 11.84)†

$$_{+}\langle v,\mathbf{s}|S|\mathbf{s}',v'\rangle_{+} = {}_{-}\langle v,\mathbf{s}|\mathbf{s}',v'\rangle_{+} \tag{5.2}$$

the S-matrix determines the transformation function from $|\mathbf{s},v\rangle_{+}$ to $|\mathbf{s}',v'\rangle_{-}$, or the probability of detecting the state $|\mathbf{s},v\rangle_{+}$ in the state $|\mathbf{s}',v'\rangle_{-}$.

From the completeness of sets $|\mathbf{s},v\rangle_{+}$ and $|\mathbf{s}',v'\rangle_{-}$, it follows that the S-matrix is unitary.

$$\begin{aligned} _{+}\langle v,\mathbf{s}|S^{+}S|\mathbf{s}',v'\rangle_{+} &= \int {}_{+}\langle v,\mathbf{s}|\mathbf{s}'',v''\rangle_{-}\,d\mathbf{s}''\,dv''\,{}_{-}\langle v'',\mathbf{s}''|\mathbf{s}',v'\rangle_{+} \\ &= \delta(\mathbf{s}-\mathbf{s}')\,\delta(v-v') \end{aligned} \tag{5.3}$$

For symmetrical wave functions the symmetrized S-matrix is to be used.

$$S_1 = \Lambda_1 S \Lambda_1, \qquad S_1 S_1{}^{+} = S_1{}^{+} S_1 = \Lambda_1 \tag{5.4}$$

Calculation gives

$$\begin{aligned} _{-}\langle v,\mathbf{s}|S_1|\mathbf{s}',v'\rangle_{-} &= {}_{+}\langle v,\mathbf{s}|S_1|\mathbf{s}',v'\rangle_{+} \\ &= \delta(\mathbf{s}-\mathbf{s}')\left\{\delta_1(v-v') + \frac{\pi i\lambda}{(2\pi)^n}\frac{(\epsilon_{\bar{k}}+\epsilon_{\bar{p}})\,\xi_2(\bar{u})\,\xi_2(\bar{u}')}{\Delta_{+}(\bar{u})}\right. \\ &\qquad \left.\times\,\delta(\omega(\bar{u})-\omega(\bar{u}'))\right\}, \\ v = \{\bar{u},l\}, &\qquad v' = \{\bar{u}',l'\} \end{aligned} \tag{5.5}$$

the S-matrix used for antisymmetric functions has no sense, because the antisymmetry property of the wave function is not conserved in transformation from basis $|s,v\rangle_{+}$ to basis $|\mathbf{s},v\rangle_{-}$, while the symmetry property is conserved, due to equivalence of (3.43) and (3.44).

6. *Concluding Remarks*

Using a quantization method based on the separation of concepts of energy and Hamiltonian we have succeeded in separating quantization problems into parts. The exact solutions for the one-emlon and two-emlon cases were obtained. The typical difficulties for the traditional approach, such as the vacuum non-stationarity and the infinite density fluctuations at the vacuum state, do not exist. These difficulties are noted by Dirac (1958) as grave disadvantages of the relativistic quantum theory.

† In non-relativistic quantum mechanics the S-matrix defined by the indicated relation is also a scattering matrix. It may be shown using the fact that in this case an operator of the particle number density has the form $\psi^{+}(x)\,\psi(x)$. The question as to whether S-matrix defined by (5.2) is a scattering matrix at this stage is left open, because there is no definition of the particles density. I believe that it is ambiguous to suppose that the S-matrix is the scattering matrix according to this definition, because it may be that no reasonable operator of the particles number density corresponds to such a matrix.

The suggested method is logically more consistent than the traditional one. It contains no new principles as compared to non-relativistic quantum mechanics. For example, a concept of normal product is not used, the relation (1.2) characteristic for non-relativistic quantum mechanics is used instead of it.

Though a model with interaction Lagrangian $(\lambda/2)\varphi^+\varphi^+\varphi\varphi$ was considered, the obtained results are true for interaction Lagrangian of the form $f(\varphi^+\varphi)$, where $f(x)$ is an arbitrary, rather smooth, function which has the property $f(0)=0$. The reason is that in the considered one-emlon and two-emlon problems only the initial two expansion terms of $f(\varphi^+\varphi)$, $f_1\varphi^+\varphi$ and $f_2\varphi^+\varphi^+\varphi\varphi$ are essential. The rest terms do not make a contribution.

Appendix

Let us calculate the integral

$$I_n = \frac{1}{(2\pi)^n}\int \frac{\xi_1(u)\,\xi_2(u)}{\omega-\omega(u)}\,du \tag{A.1}$$

where n is the dimension of configurational space. In the developed form it is

$$I_n = \frac{1}{(4\pi)^n}\sum_{\epsilon_k,\epsilon_p=\pm 1}\int \frac{\epsilon_k\,d\mathbf{q}}{4E_1E_2(\omega-\epsilon_kE_1-\epsilon_pE_2)}$$

$$E_1 = \sqrt{\left[m^2+\left(\frac{s+q}{2}\right)^2\right]},\qquad E_2=\sqrt{\left[m^2+\left(\frac{s-q}{2}\right)^2\right]} \tag{A.2}$$

$$I_n = -\frac{2}{(4\pi)^n}\int \frac{(\mathbf{sq}-\omega^2)\,d\mathbf{q}}{[\omega^4-\omega^2(4m^2+s^2+q^2)-(\mathbf{sq})^2]\sqrt{[4m^2+(\mathbf{s}-\mathbf{q})^2]}} \tag{A.3}$$

From (A.3) it follows that I_n depends only on parameters m^2, ω^2 and $\mathbf{s}$.

For the calculation of integral (A.3), let us choose a coordinate system so that axis q_1 has direction of vector $\mathbf{s}$. Then, for example,

$$I_2(m^2,\omega^2,\mathbf{s}) = -\frac{2}{(4\pi)^2}\int_{-\infty}^{\infty} dq_2$$

$$\times\int_{-\infty}^{\infty}\frac{(sq_1-\omega^2)\,dq_1}{[\omega^4-\omega^2(4m^2+q_2^2+s^2+q_1^2)-(sq_1)^2]\sqrt{(4m^2+q_2^2+s^2-2sq_1+q_1^2)}}$$

$$= \frac{1}{4\pi}\int_{-\infty}^{\infty} dq_2\, I_1\left(m^2+\frac{q_1^2}{4}\omega^2,\mathbf{s}\right),\qquad s=|\mathbf{s}| \tag{A.4}$$

By analogy

$$I_n(m^2, \omega^2, s) = \frac{1}{4\pi} \int_{-\infty}^{\infty} dq_n I_{n-1}\left(m^2 + \frac{q_n^2}{4}, \omega^2, s\right) \tag{A.5}$$

As a matter of fact, I_1 will be shown to depend only on parameters m^2 and $\gamma^2 = \omega^2 - s^2$. Then from (A.5) it follows that for any n, I_n (if it exists) depends only on m^2 and γ^2 and in the calculation of (A.3), one can assume that $s = 0$, and then $\gamma^2 = \omega^2$. It essentially simplifies the calculation of the integral.

Thus, the problem is reduced to the calculation of the integral I_1, which is considered as a function of a complex variable $z = \omega^2$

$$I_1(z, s) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{dq(sq - z)}{(\gamma^2 q^2 + z\xi^2)\sqrt{[4m^2 + (s-q)^2]}}$$

$$\gamma^2 = z - s^2, \qquad \xi^2 = 4m^2 - \gamma^2 \tag{A.6}$$

The integrand which is considered as a function of the complex variable q has the following singularities: two branching points, $q = s \pm 2im$, and two poles

$$q = i\frac{\xi}{\gamma}\sqrt{z} = \sqrt{\left[\frac{z(z - 4m^2 - s^2)}{z - s^2}\right]} \tag{A.7}$$

The poles lie at the imaginary axis if $s^2 < z < 4m^2 + s^2$; they lie at the real axis, i.e. at the integration path, if $4m^2 + s^2 \leqslant z$ or $0 < z \leqslant s^2$. In this case (A.6) depends on the way of the pole bypass. Let us consider $I_1(z,s)$ as a function in complex plane z with the cuts along the real axis $(-\infty, s^2]$ and $[4m^2 + s^2, \infty)$. We have the different values of $I_1(z,s)$ on the upper and lower edges of cuts. A simple analysis shows that if the point z motion trajectory is in the finite part of complex plane of z and does not cross the cuts, then the corresponding motion trajectories of the poles (A.7) do not cross the integration path. Let us calculate (A.6) for $s^2 < z < 4m^2 + s^2$.

For this purpose, let us make the substitution of the integration variable

$$q = 2m\,\mathrm{sh}\varphi + s \tag{A.8}$$

and introduce the designations

$$\gamma = \sqrt{(\omega^2 - s^2)}, \qquad \xi = \sqrt{(4m^2 - \gamma^2)}, \qquad \alpha = \omega\xi/\gamma,$$
$$A = (2\pi)^{-1}, \qquad y = s + i\alpha \tag{A.9}$$

where γ, ξ, α, ω^2, s are real values, and y is a complex one.

One has for (A.6)

$$I_1(\omega, s) = A(j + j^*) \tag{A.10}$$

where (*) means a complex conjugate and

$$j = \frac{\xi s - i\omega\gamma}{2\xi\gamma^2} \int\limits_{-\infty}^{\infty} \frac{d\varphi}{2m\,\mathrm{sh}\varphi + y} \tag{A.11}$$

One gets after integration of (A.11)

$$j = \frac{s\xi - i\omega\gamma}{2\xi\gamma^2\sqrt{(4m^2+y^2)}} \ln \left. \frac{y\,\mathrm{th}\frac{\varphi}{2} - 2m + \sqrt{(4m^2+y^2)}}{y\,\mathrm{th}\frac{\varphi}{2} - 2m - \sqrt{(4m^2+y^2)}} \right|_{\varphi=-\infty}^{\varphi=+\infty} \tag{A.12}$$

Let us take into account that according to (A.9)

$$4m^2 + y^2 = -\frac{1}{\gamma^2}(\xi s - i\omega\gamma)^2 \tag{A.13}$$

Let us substitute (A.12) and the corresponding complex conjugate expression into (A.10) and make the necessary calculations. As a result one obtains

$$I_1(\omega^2, s) = I_1(\gamma^2) = -\frac{1}{\pi\xi\gamma}\,\mathrm{arctg}\frac{\gamma}{\xi}, \qquad 0 < \gamma^2 < 4m^2 \tag{A.14}$$

where arctg is the principal value of arctangent.

Making the expansion in series of (A.14) one gets

$$I_1(\gamma^2) = -\frac{1}{\pi}\left\{\frac{1}{4m^2-\gamma^2} - \frac{1}{3}\frac{\gamma^2}{(4m^2-\gamma^2)^2} + \frac{1}{5}\frac{\gamma^4}{(4m^2-\gamma^2)^3} - \cdots\right\} \tag{A.15}$$

From (A.14) and (A.15) it follows that I_1 is a function only of γ^2. The I_1 is analytical in the complex plane γ^2 with the cut along real axis $[4m^2, \infty)$. The cut $(-\infty, s^2]$ which was made in the complex plane ω^2 and the corresponding cut $(-\infty, 0]$ in the plane γ^2 are unessential, because $I_1(\gamma^2)$ takes the same value on their edges.

The values of $I_1(\gamma^2)$ for other values of γ^2 may be obtained making analytical prolongation of (A.14) to the appropriate region.

In particular, assuming that $\gamma_0 = |\gamma|$, $\xi_0 = |\xi|$, one gets for real values of γ^2

$$\mathrm{Re}\, I_1 = \begin{cases} -\dfrac{1}{2\pi\xi_0\gamma_0}\ln\left|\dfrac{\xi_0+\gamma_0}{\xi_0-\gamma_0}\right|, & \gamma^2 < 0 \\ -\dfrac{1}{\pi\xi_0\gamma_0}\,\mathrm{arctg}\dfrac{\gamma_0}{\xi_0}, & 0 < \gamma^2 < 4m^2 \\ \dfrac{1}{2\pi\xi_0\gamma_0}\ln\left|\dfrac{\gamma_0+\xi_0}{\gamma_0-\xi_0}\right|, & 4m^2 < \gamma^2 \end{cases} \tag{A.16}$$

$$\mathrm{Im}\, I_1 = \begin{cases} 0, & \gamma^2 < 4m^2 \\ -(2\xi_0\gamma_0)^{-1}, & \gamma^2 = \gamma_0{}^2 + i\delta, \quad \gamma_0{}^2 > 4m^2, \quad \delta \to +0 \\ (2\xi_0\gamma_0)^{-1}, & \gamma^2 = \gamma_0{}^2 - i\delta, \quad \gamma_0{}^2 > 4m^2, \quad \delta \to +0 \end{cases}$$

Let us consider I_1 as a function of the complex variable ω. I_1 will be an analytical function of ω in the complex plane with the cuts along real axis $(-\infty,-|4m^2+s^2|^{1/2}]$, $[|4m^2+s^2|^{1/2},\infty)$. Let us suppose that

$$I^{(\pm)}(\omega)=\lim_{\delta\to+0} I_1(\omega\pm i\delta)=\lim_{\delta\to+0}\int\frac{\xi_1(u)\,\xi_2(u)\,du}{\omega-\omega(u)\pm i\delta} \tag{A.17}$$

then, according to (A.16), one has for $n=1$

$$I^{(\pm)}(\omega)=I^{(\mp)}(-\omega) \tag{A.18}$$

$$I^{+}(\omega)-I^{(-)}(\omega)=\begin{cases}-\dfrac{i}{|\gamma^2(\gamma^2-4m^2)|^{1/2}}, & \omega>\sqrt{(4m^2+s^2)}\\ 0, & 0<\omega<\sqrt{(4m^2+s^2)}\end{cases} \tag{A.19}$$

According to (A.5), I_2 is a function of γ^2 which may be calculated for $0<\gamma^2<4m^2$ putting in (A.3) $n=2$, $s=0$, $\gamma=\omega$, $\xi=\sqrt{(4m^2-\gamma^2)}$:

$$\begin{aligned}I_2(\gamma^2)&=-\frac{1}{4\pi}\int_0^\infty\frac{q\,dq}{(q^2+\xi^2)\sqrt{(4m^2+q^2)}}=-\frac{1}{8\pi}\int_0^\infty\frac{dz}{(z+\xi^2)\sqrt{(z+4m^2)}}\\&=-\frac{1}{8\pi\gamma}\ln\frac{2m+\gamma}{2m-\gamma}\end{aligned} \tag{A.20}$$

(A.20) defined $I_2(\gamma^2)$ as a function of γ^2 analytical in the whole complex plane with the cut along real axis $[4m^2,\infty)$.

Finally, for $n=3$ (A.3) diverges. Hence, strictly speaking (A.5) is not valid for $n=3$. Nevertheless, we consider that (A.3) defines I_3 as a function of γ^2 then one can try to determine it substituting into (A.3) $s=0$, $\gamma^2=\omega^2$ and integrating not over the whole space of $\mathbf{q}$, but over the sphere of radius R and then tending R to infinity. One gets

$$\begin{aligned}I_3(\gamma^2)&=\frac{1}{(4\pi)^2}\lim_{R\to\infty}\int_0^R q^2\,dq\int_0^R\frac{\sin\vartheta\,d\vartheta}{(\xi^2+q^2)\sqrt{(4m^2+q^2)}}\\&=-\frac{2}{(4\pi)^2}\left\{-2\pi\xi^2 I_1(\gamma^2)+\lim_{R\to\infty}\int_0^R\frac{dq}{\sqrt{(4m^2+q^2)}}\right\}\\&=\frac{\xi^2 I_1(\gamma^2)}{4\pi}-\lim_{R\to\infty}\frac{1}{8\pi^2}\ln\left[\frac{R}{2m}+\sqrt{\left(\frac{R^2}{4m^2}+1\right)}\right]\end{aligned} \tag{A.21}$$

Thus $I_3(\gamma^2)$ represents a sum of the infinite constant and function of γ^2, analytical in the complex plane γ^2 with the cut along the real axis $[4m^2,\infty)$. Being considered as a function of ω, I_3 represents a sum of infinite constant and a function analytical in the complex plane ω with the cuts $(-\infty,$

$-|4m^2+s^2|^{1/2}]$, $[|4m^2+s^2|^{1/2},\infty)$ along the real axis. The jump of $I_3(\omega)$ at the cut edges is finite

$$I_3^{(+)}(\omega)-I_3^{(-)}(\omega)=\begin{cases}-i(4\pi)^{-1}|\gamma|^{-1/2}|\gamma^2-4m^2|^{1/2}, & \omega<-|4m^2+s^2|^{1/2}\\ 0, & -|4m^2+s^2|^{1/2}<\omega<|4m^2+s^2|^{1/2}\\ i(4\pi)^{-1}|\gamma|^{-1/2}|\gamma^2-4m^2|^{1/2}, & |4m^2+s^2|^{1/2}<\omega\end{cases}$$

The divergence of $I_3(\omega)$ itself leads to no difficulties, because in the functions (3.13) and (3.14) $I(\omega)$ enters the denominator.

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